ON PRINCIPAL MINORS OF POSITIVE DEFINITE MATRICES

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Abstract

Simple arguments are given for the inductive proof of the well-known result that all principal leading minors of a (symmetric) positive definite matrix are positive.

It is well known that all principal leading minors of a positive definite (p.d.) matrix (assumed, as is customary, to be symmetric) are positive. Seelye (1958) gives an inductive proof, for the bulk of which we here offer simpler arguments.

Theorem: A real symmetric matrix is positive definite if and only if all its principal leading minors are positive.

Proof: Let $\underset{\sim}{A} = \underset{\sim}{A}'$ be real and p.d., and for vectors $\underset{\sim}{b}$ and $\underset{\sim}{x}$ and scalars β and λ define

$$\underset{\sim}{B} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{b} \\ \underset{\sim}{b}' & \beta \end{bmatrix} \text{ and } \underset{\sim}{w} = \begin{bmatrix} \underset{\sim}{x} \\ \lambda \end{bmatrix} \tag{1}$$

where $\underset{\sim}{x}$ and $\underset{\sim}{A}$ have the same number of rows. The inductive proof is based on assuming the positive definiteness of $\underset{\sim}{A}$ and showing that $\underset{\sim}{B}$ is then p.d. if and only if all principal leading minors of $\underset{\sim}{B}$ are positive. Because these minors consist of all such minors of $\underset{\sim}{A}$, and $|\underset{\sim}{B}|$ itself, and because $\underset{\sim}{A}$ is being assumed p.d., we then have only to show that $\underset{\sim}{B}$ is p.d. if and only if $|\underset{\sim}{B}|$ is positive. To do this we use

$$|\underset{\sim}{B}| = |\underset{\sim}{A}| \; (\beta - \underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}) \tag{2}$$

and

$$\underset{\sim}{w}'\underset{\sim}{B}\underset{\sim}{w} = \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} + 2\underset{\sim}{b}'\underset{\sim}{x}\lambda + \beta\lambda^2 \tag{3}$$

$$= \beta(\lambda + \underset{\sim}{b}'\underset{\sim}{x}/\beta)^2 + \underset{\sim}{x}'(\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{b}'/\beta)\underset{\sim}{x} \; . \tag{4}$$

Proof (i): that if $\underset{\sim}{B}$ is p.d., then $|\underset{\sim}{B}| > 0$.

$\underset{\sim}{B}$ being p.d. means that $\underset{\sim}{w}'\underset{\sim}{B}\underset{\sim}{w} > 0$ for all $\underset{\sim}{w} \neq \underset{\sim}{0}$. First take $\underset{\sim}{w}' = [\underset{\sim}{0} \quad \lambda]$, i.e., from (1), $\underset{\sim}{x} = \underset{\sim}{0}$. Then from (4), $\underset{\sim}{w}'\underset{\sim}{B}\underset{\sim}{w} > 0 \Rightarrow \beta\lambda^2 > 0 \Rightarrow \beta > 0$ because λ is real. Second, take $\underset{\sim}{w}' = [\underset{\sim}{b}'\underset{\sim}{A}^{-1} \; - \underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}/\beta]$; i.e., $\underset{\sim}{x}' = \underset{\sim}{b}'\underset{\sim}{A}^{-1}$ and $\lambda = -\underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}/\beta$. Then again from (4)

$$\underset{\sim}{w}'\underset{\sim}{B}\underset{\sim}{w} > 0 \Rightarrow \underset{\sim}{b}'\underset{\sim}{A}^{-1}(\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{b}'/\beta)\underset{\sim}{A}^{-1}\underset{\sim}{b} > 0$$

i.e.

$$\underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}(1 - \underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}/\beta) > 0 \; . \tag{5}$$

Because $\underset{\sim}{A}$ is p.d., $|\underset{\sim}{A}| > 0$; and $\underset{\sim}{A}^{-1}$ is also p.d., and so $\underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b} > 0$. Therefore from (5) we have $\beta - \underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b} > 0$ and this together with $|\underset{\sim}{A}| > 0$ implies from (2) that $|\underset{\sim}{B}| > 0$. Q.E.D.

Proof (ii): that if $|\underset{\sim}{B}| > 0$, then $\underset{\sim}{B}$ is p.d.

$|\underset{\sim}{B}| > 0$, along with $|\underset{\sim}{A}| > 0$ implies from (2) that $\beta > \underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}$. Then in (3)

$$\begin{aligned}\underset{\sim}{w}'\underset{\sim}{B}\underset{\sim}{w} &> \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} + 2\lambda\underset{\sim}{b}'\underset{\sim}{x} + \lambda^2(\underset{\sim}{b}'\underset{\sim}{A}^{-1}\underset{\sim}{b}) \\ &> (\underset{\sim}{x} + \lambda\underset{\sim}{A}^{-1}\underset{\sim}{b})'\underset{\sim}{A}(\underset{\sim}{x} + \lambda\underset{\sim}{A}^{-1}\underset{\sim}{b}) \\ &> 0 \text{ because } \underset{\sim}{A} \text{ is p.d.}\end{aligned}$$

Therefore $\underset{\sim}{B}$ is p.d. Q.E.D.

As final comment, we note that although the theorem is in terms of principal leading minors it applies to all principal minors. We state this as a corollary.

Corollary: All principal minors of a symmetric p.d. matrix are positive.

Proof: Let $\underset{\sim}{A} = \underset{\sim}{A}'$ be p.d.. Then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} > 0$ for all $\underset{\sim}{x} \neq \underset{\sim}{0}$ and, for $\underset{\sim}{P}$ being a permutation matrix, $\underset{\sim}{x}'\underset{\sim}{P}'\underset{\sim}{A}\underset{\sim}{P}\underset{\sim}{x} > 0$ also. Therefore $\underset{\sim}{P}'\underset{\sim}{A}\underset{\sim}{P}$ is p.d., and so by the theorem its principal leading minors are positive. But, over the set of all possible $\underset{\sim}{P}$'s, the principal leading minors of $\underset{\sim}{P}'\underset{\sim}{A}\underset{\sim}{P}$ are the principal minors of $\underset{\sim}{A}$, which are therefore positive. Q.E.D.

Reference

Seelye, C. J. (1958) Conditions for a positive difinite quadratic form established by induction. Am. Mathematical Monthly, 65, 355-6.